

Network Multibaker Maps, Information Theory and Stochastic Thermodynamics

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Stochastic thermodynamics (ST) summarizes recent advances in nonequilibrium statistical mechanics of systems featuring local thermodynamic equilibrium. However, the mathematical description of ST is largely independent of this thermodynamic context. Here, we introduce novel network multibaker maps as a model system for reversible, chaotic dynamics. As a result, the mathematical formalism of ST emerges from information theory, and is shown to be consistent with previous work.

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Introduction Stochastic thermodynamics (ST) is a recent approach to the statistical mechanics of nonequilibrium systems [1]. Using both random variables and ensemble properties it extends the notions of entropy and entropy production to stochastic trajectories [2, 3]. Using the notion of time-reversal, ST provides elegant derivations [4] of stochastic fluctuation [5, 6] and work relations [7, 8]. It also yields universal results on thermodynamic efficiency of small machines [9] and the conversion of information to work [10, 11]. Moreover, advanced experimental and simulation techniques allow direct testing of these theoretical results [12–14]. ST relies on a thermodynamically consistent interpretation of the stochastic process within a framework of an assumed thermodynamic *local equilibrium*. Surprisingly, the mathematical formulation of ST seems not to depend on the specific physical context [15, 16]. What is needed is the *Markov property* of the dynamics, though for certain relations even this assumptions can be relaxed [17].

In this work, we introduce the notion of *network multibaker maps* (NMBM) and take a dynamical system's perspective on ST. Multibaker maps serve as a paradigm for studies on the relation of microscopic dynamics, statistical physics and transport phenomena [18, 19]. They fulfill many of the generic features of chaotic dynamics: Reversibility, hyperbolicity, homo- and heteroclinic orbits, a natural Markov partition and ergodicity [20, 21]. Yet, their simple structure admits an explicit calculation of the evolution of ensembles which are represented by non-stationary phase-space densities [18, 22]. Here, we explicitly calculate the full evolution of information-theoretical quantities for NMBM. We distinguish three fundamental contributions, which are in one-to-one correspondence to the entropy expressions used in ST. Further, we show that our approach is consistent with previous work based on phase-space contraction [23, 24].

Network multibaker maps Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected graph with N vertices $i \in \mathcal{V} := [1, 2, \dots, N]$ and edges $e \in \mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Vertices represent the fundamental states of some effective (for instance, thermodynamic) description. Edges indicate possible transitions between these states. We assume bi-directional edges,

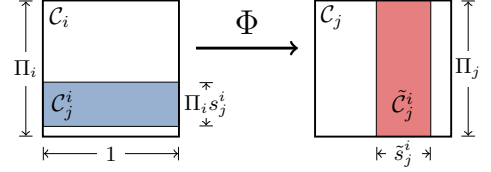


FIG. 1. Dynamics of the network multibaker map. For each of the k_i neighbours vertices $j \in \mathcal{V}_i$ of vertex i , a horizontal strip $C_j^i \subset C_i$ of relative height s_j^i is affine-linearly mapped to a vertical strip $\tilde{C}_j^i \subset C_j$ of width \tilde{s}_j^i .

i.e. $e := (i, j) \in \mathcal{E} \Leftrightarrow \bar{e} := (j, i) \in \mathcal{E}$, and that each vertex is connected to itself, i.e. $(i, i) \in \mathcal{E}, \forall i \in \mathcal{V}$. Let $\mathcal{V}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ be the set of vertices that are connected to a state i . Denote by $k_i := |\mathcal{V}_i|$ the degree of vertex i . Associate with each vertex i a rectangular *cell* $C_i := [0, 1] \times \Pi_i \cdot [0, 1]$ with area (Lebesgue measure) Π_i consisting of the set of phase-space points $x \in C_i$ which belong to state i . The overall *phase space* Γ of the system is the disjoint union $\Gamma := \cup_{i=1}^N C_i$.

A NMBM, $\Phi : \Gamma \rightarrow \Gamma$, deterministically maps phase-space points to adjacent cells. It is specified geometrically (cf. Fig. 1) by dividing each cell C_i into k_i horizontal strips of finite relative height $s_j^i > 0$. More precisely, with the offset $b_j^i = \sum_{k < j} s_k^i$:

$$C_j^i := \begin{cases} [0, 1] \times \Pi_i \cdot [b_j^i, b_j^i + s_j^i] & i \in \mathcal{V}, j \in \mathcal{V}_i, \\ \emptyset & \text{else.} \end{cases} \quad (1)$$

The dynamics maps each horizontal strip $C_j^i \subset C_i$ to a vertical strip $\tilde{C}_j^i := \Phi C_j^i \subset C_j$ in an affine-linear way:

$$\tilde{C}_j^i := [\tilde{b}_j^i, \tilde{b}_j^i + \tilde{s}_j^i] \times \Pi_j \cdot [0, 1] \quad (2)$$

Analogously, \tilde{s}_j^i denote the relative width of the vertical strips and the offsets read $\tilde{b}_j^i = \sum_{k < i} \tilde{s}_k^i$. Points in any strip are mapped affine-linearly, such that the horizontal direction is contracted by a factor $s_j^i < 1$ whereas the vertical direction is expanded by a factor $(s_j^i)^{-1} > 1$. The multibaker map is fully defined by specifying \mathcal{G} and the numbers Π_i, s_j^i and \tilde{s}_j^i .

In deriving fluctuation relations in both stochastic and deterministic dynamics, one applies a natural notion of

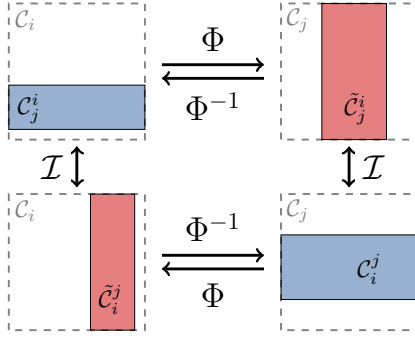


FIG. 2. The multibaker dynamics feature a volume preserving involution \mathcal{I} , cf. Eq. (5), which interchanges horizontal and vertical stripes in one cell, *i.e.* $\mathcal{I}\tilde{\mathcal{C}}_j^i = \mathcal{C}_i^j$. The dynamics Φ is reversible, but generally not volume preserving.

time-reversal [3, 6, 23, 25] which has its origins in the time-reversibility of fundamental physical laws. Following Ref. [26] we are interested in dynamics with a transformation \mathcal{I} on Γ which a) is an involution, b) preserves phase-space volume $|\cdot|$ and c) facilitates time-reversal:

$$\mathcal{I} \circ \mathcal{I} = \text{id}, \quad (3a)$$

$$|\mathcal{I}^{-1}A| = |A|, \quad \forall A \subset \Gamma, \quad (3b)$$

$$\Phi^{-1}x = (\mathcal{I} \circ \Phi \circ \mathcal{I})x, \quad \forall x \in \Gamma. \quad (3c)$$

Properly thermostatted multibaker maps [26] obey

$$\tilde{s}_j^i = s_i^j, \quad \forall i, j, \quad (4)$$

such that Eq. (3) is fulfilled by the involution [18]

$$\mathcal{I} : (x, y) \mapsto (1 - \Pi_i^{-1}y, \Pi_i(1 - x)), \quad \forall (x, y) \in \mathcal{C}_i, \quad (5)$$

which additionally is local each cell (*cf.* Fig. 2), *i.e.*

$$\mathcal{I}\mathcal{C}_i = \mathcal{C}_i, \quad \forall i \quad (6)$$

With the general definition $\tilde{\mathcal{C}}_j^i := \Phi\mathcal{C}_i \cap \mathcal{C}_j$, Eq. (4) follows solely from Eqs. (3) and (6), which ensure that $\tilde{\mathcal{C}}_j^i$ and \mathcal{C}_i^j are interchanged by the involution (*cf.* Fig. 2).

Densities and trajectories In statistical physics, the probability of finding a system at some point $x \in \Gamma$ is given by a time-dependent ensemble measure. We assume that this ensemble is absolutely continuous with respect to the Lebesgue measure, *i.e.* it is characterized by a phase-space density $\varrho^{(\nu)}(x)$, where the superscript (ν) indicates discrete time. This phase-space density evolves according to Φ and generically develops an intricate structure as the number of iterations increases. Practically, preparation and resolution of phase-space densities is restricted to a coarse-grained level, which is represented by the vertices of \mathcal{G} . Hence, we define a coarse-grained density $\rho^{(\nu)}(x)$ to take the average value $\rho_i^{(\nu)}$ of $\varrho^{(\nu)}$ on cell \mathcal{C}_i :

$$\rho_i^{(\nu)} := \Pi_i^{-1} \int_{\mathcal{C}_i} \varrho^{(\nu)}(x) dx, \quad (7a)$$

$$\rho^{(\nu)}(x) := \sum_i \chi_i(x) \rho_i^{(\nu)}, \quad (7b)$$

where χ_i is the characteristic or indicator function of \mathcal{C}_i .

Due to the preparation procedure, fine- and coarse-grained ensembles agree initially, *i.e.* $\varrho^{(0)}(x) \equiv \rho^{(0)}(x)$. To follow the evolution of $\varrho^{(0)}$ to $\varrho^{(\nu)}$ we introduce trajectories $\underline{\omega}$ as $(\nu + 1)$ -tuples with entries $\omega_k \in \mathcal{V}$:

$$\underline{\omega} := (\omega_0, \omega_1, \dots, \omega_\nu). \quad (8)$$

The entry ω_k denotes the vertex visited at time k . Hence, a trajectory is a finite string of a the symbolic sequence associated with the (natural) Markov partition [27].

Let the set $\tilde{\mathcal{C}}[\underline{\omega}]$ denote all phase-space points that move together along a fixed trajectory $\underline{\omega}$ and hence reside in cell \mathcal{C}_{ω_ν} at time ν . Denoting by $\underline{\omega}^{(k)}$ the first $k \leq \nu$ steps of $\underline{\omega}$, we define recursively:

$$\tilde{\mathcal{C}}[\underline{\omega}^{(k)}] := \Phi(\tilde{\mathcal{C}}[\underline{\omega}^{(k-1)}] \cap \mathcal{C}_{\omega_k}^{\omega_{k-1}}), \quad \forall 0 < k \leq \nu, \quad (9)$$

where $\tilde{\mathcal{C}}[\underline{\omega}^{(0)}] := \mathcal{C}_{\omega_0}$. One can easily verify that $\tilde{\mathcal{C}}[\underline{\omega}^{(k-1)}]$ is a vertical strip. Intersecting with the horizontal strip $\mathcal{C}_{\omega_k}^{\omega_{k-1}}$ reduces its volume by a factor $s_{\omega_k}^{\omega_{k-1}} < 1$. Through contraction and expansion in horizontal and vertical directions, respectively, Φ changes the volume by another factor $\tilde{s}_{\omega_k}^{\omega_{k-1}}/s_{\omega_k}^{\omega_{k-1}}$. Hence, $|\mathcal{C}[\underline{\omega}^{(k)}]| = \tilde{s}_{\omega_k}^{\omega_{k-1}} |\mathcal{C}[\underline{\omega}^{(k-1)}]|$ and after iteration to $k = \nu$,

$$|\mathcal{C}[\underline{\omega}^{(\nu)}]| = \Pi_{\omega_\nu} \prod_{k=1}^{\nu} \tilde{s}_{\omega_k}^{\omega_{k-1}} = \Pi_{\omega_\nu} \prod_{k=1}^{\nu} s_{\omega_{k-1}}^{\omega_k}, \quad (10)$$

where for the last equality we used Eq. (4).

Because the initial microscopic density $\varrho^{(0)}$ is uniform on each cell, so is the density on $\tilde{\mathcal{C}}[\underline{\omega}]$ for any $\underline{\omega}$. Denote by $\varrho[\underline{\omega}; k]$ the density on the $(\nu - k)$ th pre-image $\Phi^{\nu-k}\tilde{\mathcal{C}}[\underline{\omega}]$. Probability conservation and uniform contraction imply

$$\varrho[\underline{\omega}; k] = (\eta_{\omega_k}^{\omega_{k-1}})^{-1} \varrho[\underline{\omega}; k-1] \quad (11)$$

where we introduced the contraction factor

$$\eta_{\omega_k}^{\omega_{k-1}} := \frac{|\tilde{\mathcal{C}}_{\omega_k}^{\omega_{k-1}}|}{|\mathcal{C}_{\omega_k}^{\omega_{k-1}}|} \equiv \frac{\Pi_{\omega_k} \tilde{s}_{\omega_k}^{\omega_{k-1}}}{\Pi_{\omega_{k-1}} s_{\omega_k}^{\omega_{k-1}}}. \quad (12)$$

Iterating this condition, we find with Eq. (4) that the density $\varrho^{(\nu)}(x)$ at a point x with history $\underline{\omega}(x)$ obeys:

$$\varrho^{(\nu)}(x) = \varrho[\underline{\omega}(x); \nu] = \rho_{\omega_0} \prod_{k=1}^{\nu} \left[\frac{s_{\omega_k}^{\omega_{k-1}}}{s_{\omega_{k-1}}^{\omega_k}} \right] \frac{\Pi_{\omega_0}}{\Pi_{\omega_\nu}}. \quad (13)$$

Integrating $\varrho(x)$ over a cell Π_i yields the probability

$$p_i^{(k)} := \int_{\mathcal{C}_i} \varrho^{(k)}(x) dx = \Pi_i \rho_i^{(k)}, \quad (14)$$

which evolves according to the *Master equation* [27]

$$p_i^{(k)} = \sum_j \left[p_j^{(k-1)} s_i^j \right]. \quad (15)$$

This establishes the connection to *Markov chains* and hence the (discrete) formulation of ST [4].

Information and entropy The Shannon entropy [28] of a probability density $\varrho(x)$ on Γ is defined as

$$\mathcal{S}[\varrho] := - \int_{\Gamma} \varrho \log \varrho \, dx. \quad (16)$$

With the results (10) and (13) we can calculate the fine-grained entropy of $\varrho^{(\nu)}(x)$ as a sum over all trajectories $\underline{\omega}^{(\nu)}$ of length ν weighted with $|\mathcal{C}[\underline{\omega}^{(\nu)}]|$

$$\begin{aligned} S_{\text{fg}}^{(\nu)} &:= \mathcal{S}[\varrho^{(\nu)}] = - \sum_{\underline{\omega}^{(\nu)}} \left[|\mathcal{C}[\underline{\omega}^{(\nu)}]| \times (\varrho[\underline{\omega}; \nu] \log \varrho[\underline{\omega}; \nu]) \right] \\ &= \sum_{\underline{\omega}^{(\nu)}} \left[\mathbb{P}[\underline{\omega}] \left(\log \Pi_{\omega_{\nu}} - \log p_{\omega_0}^{(0)} - \sum_{k=1}^{\nu} B_{\omega_k}^{\omega_{k-1}} \right) \right] \end{aligned} \quad (17)$$

where

$$\mathbb{P}[\underline{\omega}^{(\nu)}] \equiv |\mathcal{C}[\underline{\omega}^{(\nu)}]| \varrho[\underline{\omega}; \nu] = p_{\omega_0}^{(0)} \prod_{k=1}^{\nu} s_{\omega_k}^{\omega_{k-1}}, \quad (18)$$

$$B_j^i := \log \left(s_j^i / s_i^j \right). \quad (19)$$

Similarly, we find

$$S_{\text{cg}}^{(\nu)} := \mathcal{S}[\varrho^{(\nu)}] = \sum_{\underline{\omega}^{(\nu)}} \left[\mathbb{P}[\underline{\omega}^{(\nu)}] \left(\log \Pi_{\omega_{\nu}} - \log p_{\omega_{\nu}}^{(\nu)} \right) \right]. \quad (20)$$

Trajectory functionals The forms of Eqs. (17) and (20) suggest to write them as weighted averages of trajectory functionals $s[\underline{\omega}; \nu]$, i.e.:

$$\langle\langle s \rangle\rangle^{(\nu)} = \sum_{\underline{\omega}^{(\nu)}} \left[\mathbb{P}[\underline{\omega}] s[\underline{\omega}; \nu] \right] \quad (21)$$

With that we can write $S_{\text{fg}}^{(\nu)} = \langle\langle s_{\text{fg}} \rangle\rangle^{(\nu)}$ and $S_{\text{cg}}^{(\nu)} = \langle\langle s_{\text{cg}} \rangle\rangle^{(\nu)}$, where the functionals

$$s_{\text{fg}}[\underline{\omega}, \nu] = s_{\text{int}}[\underline{\omega}, \nu] + s_{\text{vis}}[\underline{\omega}, 0] - s_{\text{mot}}[\underline{\omega}, \nu], \quad (22a)$$

$$s_{\text{cg}}[\underline{\omega}, \nu] = s_{\text{int}}[\underline{\omega}, \nu] + s_{\text{vis}}[\underline{\omega}, \nu], \quad (22b)$$

are linear combinations of three fundamental functionals:

$$s_{\text{int}}[\underline{\omega}, \nu] = s_{\text{int}}(\omega_{\nu}) = \log \Pi_{\omega_{\nu}}, \quad (23a)$$

$$s_{\text{vis}}[\underline{\omega}, \nu] = s_{\text{vis}}(\omega_{\nu}, \nu) = -\log p_{\omega_{\nu}}^{(\nu)}, \quad (23b)$$

$$s_{\text{mot}}[\underline{\omega}, \nu] = s_{\text{mot}}[\underline{\omega}] = \sum_{k=1}^{\nu} B_{\omega_k}^{\omega_{k-1}}. \quad (23c)$$

Note that the functionals s_{int} and s_{vis} depend on the final state ω_{ν} of the trajectory only, i.e. $s[\underline{\omega}; \nu] = s(\omega_{\nu}; \nu)$. Hence, we say they have a *local form*. Trajectory averages of local forms reduce to ensemble averages *via* Eq. (15):

$$\langle\langle s \rangle\rangle^{(\nu)} = \langle s \rangle^{(\nu)} := \sum_i \left[p_i^{(\nu)} s(i; \nu) \right]. \quad (24)$$

Henceforth, we regard NMBM as a model for thermostatted reversible dynamics [29]. The Markovian description (15) for the coarse-grained ensemble on the discrete state establishes the connection to ST.

The *intrinsic entropy* s_{int} accounts for the entropy of the unobservable microscopic configurations. In the present dynamical context it is calculated *à la* Boltzmann as the logarithm of a phase-space volume (23a). In the thermodynamic context, it is the entropy of the local equilibrium ensemble which reflects the state of the environment [15]. Hence, the intrinsic entropy can be considered a “state variable” and in this respect is closest to standard thermodynamics.

The entropy that is operationally accesible in experiments is the *visible entropy* s_{vis} . It only depends on the present ensemble, i.e. the $p_i^{(\nu)}$, which can be sampled by performing many measurements on identical systems. Though it has local form and averages can be calculated with Eq. (24), it depends on the ensemble and hence is not a state variable.

The conceptually most difficult term is the third functional, s_{mot} related to the quantity B_j^i . In analogy to the electromotance for electrical circuits [30], we call B_j^i the *motance* of a transition $i \rightarrow j$. Physically, a finite motance arises from a system-environment interaction. Unlike the affinity $A_j^i = \log(p_i s_j^i / (p_j s_i^j))$, the motance B_j^i only depends on the dynamics and not on the ensemble [31]. Consequently, we call s_{mot} the *accumulated motance*.

Entropy variation and production For each fundamental functional $s[\underline{\omega}, \nu]$ there is an associated fundamental variation $\sigma[\underline{\omega}, \nu] := s[\underline{\omega}, \nu] - s[\underline{\omega}^{(\nu-1)}, \nu - 1]$. The fundamental variations

$$\sigma_{\text{int}}[\underline{\omega}, \nu] = \log \Pi_{\omega_{\nu}} - \log \Pi_{\omega_{\nu-1}}, \quad (25a)$$

$$\sigma_{\text{vis}}[\underline{\omega}, \nu] = -\log p_{\omega_{\nu}}^{(\nu)} + \log p_{\omega_{\nu-1}}^{(\nu-1)}, \quad (25b)$$

$$\sigma_{\text{mot}}[\underline{\omega}, \nu] = B_{\omega_{\nu}}^{\omega_{\nu-1}} \quad (25c)$$

can be used to construct the previously known fundamental forms of entropy variation [31]:

$$\sigma_{\text{sys}}[\underline{\omega}, \nu] = \sigma_{\text{int}}[\underline{\omega}, \nu] + \sigma_{\text{vis}}[\underline{\omega}, \nu], \quad (26a)$$

$$\sigma_{\text{tot}}[\underline{\omega}, \nu] = \sigma_{\text{vis}}[\underline{\omega}, \nu] + \sigma_{\text{mot}}[\underline{\omega}, \nu], \quad (26b)$$

$$\sigma_{\text{med}}[\underline{\omega}, \nu] = \sigma_{\text{mot}}[\underline{\omega}, \nu] - \sigma_{\text{int}}[\underline{\omega}, \nu]. \quad (26c)$$

The system entropy variation σ_{sys} is equivalent to the change of coarse-grained entropy. It consists of the visible entropy of the ensemble and the intrinsic entropy of the current state. Often, the latter is forgotten [15].

We can write the average of the *total entropy production* σ_{tot} as a Kullback-Leiber divergence,

$$\langle\langle \sigma_{\text{tot}} \rangle\rangle^{(\nu)} = \sum_{i,j} \left[p_i^{(\nu)} s_j^i \log \frac{p_i^{(\nu)} s_j^i}{p_j^{(\nu+1)} s_i^j} \right] \geq 0,$$

which is always positive. It is the temporal variation of the difference $\langle\langle s_{\text{cg}} - s_{\text{fg}} \rangle\rangle$, which itself is a Kullback-Leibler divergence. It characterizes the information gain of the microscopic over the coarse-grained density. Its positive variation (*i.e.* its monotonous growth) reflects the fact that one cannot resolve the ever-refining features of the microscopic density.

The *entropy change in the medium* σ_{med} consists of a motance and an intrinsic part. It has been understood that the latter is important for a consistent thermodynamic interpretation [4, 15]. From Eqs. (19) and (12) we see that it is directly related to phase-space contraction:

$$\sigma_{\text{med}}[\underline{\omega}] = \log \frac{s_{\omega_{\nu}}^{\omega_{\nu-1}}}{s_{\omega_{\nu-1}}^{\omega_{\nu}}} - \log \frac{\Pi_{\omega_{\nu}}}{\Pi_{\omega_{\nu-1}}} = \log \eta_{\omega_{\nu}}^{\omega_{\nu-1}} \quad (27)$$

For maps, the phase-space contraction per unit time, $\Lambda = \log J$ [32, 33], is given by the logarithm of the Jacobian determinant $J = \left| \frac{d\Phi}{dx} \right|$ of the dynamics. In the current affine-linear case, $J|_x = \eta_{\omega_k}^{\omega_{k-1}}$ for $x \in \mathcal{C}_{\omega_k}^{\omega_{k-1}}$.

Connection to the continuous description For us, NMBM serve as a model for a *stroboscopic observation* of physical dynamics with a time lag τ_{obs} . However, stochastic thermodynamics is usually formulated using *continuous-time Markov processes* rather than Markov chains. In the continuous time limit, the Master equation (15) becomes a differential equation $\dot{p}_i(t) = \sum_j W_{ij}^j p_j(t)$. Its infinitesimal generator W_{ij}^j consists of transition rates w_{ij}^j , which can be thought of as the $\tau_{\text{obs}} \rightarrow 0$ limit of transition probability divided by τ_{obs} . The continuous-time versions of the fundamental entropy and variation functionals have been identified previously [3]. They resemble the discrete-time functionals with s_j^i substituted by w_j^i and finite-time variations substituted by derivatives. However, there are some subtleties to be taken care of. Firstly, in the continuous-time description, the measure $\mathbb{P}[\underline{\omega}]$ of a finite trajectory is zero and summing over trajectories involves integration over stochastic jump times. Secondly, in the present case, the average value $\langle\langle \sigma \rangle\rangle^{(\nu)} \equiv \langle\langle s[\underline{\omega}, \nu] - s[\underline{\omega}^{(\nu-1)}, \nu-1] \rangle\rangle^{(\nu)}$ of a variation is the same as the change in one time-step of the averaged entropies, $\langle\langle s \rangle\rangle^{(\nu)} - \langle\langle s \rangle\rangle^{(\nu-1)}$. This does not apply to the continuous-time case, because averaging and taking the time-derivative do not commute for non-stationary probabilities. Hence, temporal boundary-terms need to be attributed for. Thirdly, we can rewrite the average total entropy production in the following form:

$$\langle\langle \sigma_{\text{tot}} \rangle\rangle^{(\nu)} = \sum_{i,j} \left[p_i^{(\nu)} s_j^i \log \frac{p_i^{(\nu)} s_j^i}{p_j^{(\nu)} s_i^j} \right] + \sum_i \left[p_i^{(\nu+1)} \log \frac{p_i^{(\nu)}}{p_i^{(\nu+1)}} \right]$$

Substituting s_j^i by w_j^i in the first term, one obtains the familiar term used in the continuous-time case, which is already a Kullback-Leibler divergence on its own. However, for the discrete-time case the second term is needed to account for the change of the ensemble between two jumps. It vanishes for $\tau_{\text{obs}} \rightarrow 0$ because $p_i^{(\nu)} \rightarrow p_i^{(\nu+1)}$.

Phase-space preserving systems Finally, we demonstrate the consistency of our approach in the context of phase-space preserving maps. From the *dynamical systems perspective*, we consider such maps as stroboscopic pictures of a Hamiltonian dynamics. *Thermodynamically*, a Hamiltonian system is an isolated system which does not exchange energy, particles or entropy with its environment at any time. Its steady-state distribution is *microcanonical*, *i.e.* uniform on phase space. In *stochastic terms*, the steady state of an isolated system features *detailed balance* [31] for the steady-state distribution p_i^∞ :

$$p_i^\infty s_j^i = p_j^\infty s_i^j. \quad (28)$$

Henceforth, we assume Eq. (28) and reversibility, Eq. (4), to show that the interpretations are consistent. Definition (14) directly yields $s_j^i = c_{ij} \Pi_j$ with some constant $c_{ij} = c_{ji}$. Because $\sum_j s_j^i = 1, \forall i$, it follows that $c_{ji} = 1, \forall i, j$. With p_i' denoting the iterated values of the probabilities p_i in Eq. (15), we hence find

$$\begin{aligned} \langle\langle \sigma_{\text{med}} \rangle\rangle &= \sum_{i,j} \left[p_i s_j^i \log \frac{s_j^i}{s_i^j} \right] - \sum_i [(p_i' - p_i) \log \Pi_i] \\ &= \sum_{i,j} [p_i \Pi_j \log \Pi_j - p_i \Pi_j \log \Pi_i - p_j \Pi_i \log \Pi_i] \\ &\quad + \sum_i [p_i \log \Pi_i] = \sum_i [(-p_i + p_i') \log \Pi_i] = 0. \end{aligned}$$

Hence, the average entropy change in the medium—which equals the average phase-space contraction—identically vanishes *for all times and independent of the initial ensemble*, as is expected.

Summary and outlook We have shown how for NMBM the functionals of stochastic thermodynamics emerge from information theory without relying on thermodynamic assumptions. We presented interpretations of the different notions of entropy and entropy variation, which are consistent both with information theory and work on non-phase-space preserving “thermostatted” dynamics.

In analogy to the role of Smale’s horseshoe [34] as a hallmark of chaos, NMBM could hence help understanding salient features of reversible chaotic dynamics and their connection to thermodynamics. A possible approach in that direction is to treat the NMBM vertices as linearizations of fundamental structures in such dynamical systems. Then, a coarse-graining procedure under the assumption of separation of scales, in the spirit of Ref. [16] could yield a better dynamical picture of local equilibrium.

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